Binomial distribution

Consider k measurement slots, the probability to detect a particule in a slot is p

k = number of independent slots
p = probability for one particle to be detected in one measurement slot

Mean number of detected particles : $\langle n \rangle = k.p$

1				n	1						k-n
p	p	p	p	p	1-p	1-p	1-p	1-p	1-p	1-p	1-p

Probability to have n particles detected in the n first slots and no particule detected in the k-n final slots:

$$p^n \cdot (1-p)^{k-n}$$

Probability to have n particles detected in the k slots (independant of the order):
(combination of n particules in k slots)

→ Binomial distribution $$B(n) = \binom{k}{n} p^n \cdot (1-p)^{k-n} = \frac{k.(k-1)...(k-n+1)}{n!} p^n \cdot (1-p)^{k-n}$$

Poisson distribution

We use the limits:
(Poisson distribution)

$$P(n) = \lim_{\substack{k \to \infty \\ p \to \langle n \rangle / k}} B(n)$$

- The number k of particules tends to infinity
- The probability p tends to zero
- The mean number detected is k.p

$$P(n) = \lim_{\substack{k \to \infty \\ p \to \langle n \rangle / k}} \frac{k.(k-1)\ldots(k-n+1)}{n!} p^n \cdot (1-p)^{k-n} = \lim_{\substack{k \to \infty \\ p \to \langle n \rangle / k}} \frac{1}{n!} \cdot \overbrace{\frac{k}{k}}^{1} \overbrace{\frac{(k-1)}{k}}^{1} \ldots \overbrace{\frac{(k-n+1)}{k}}^{1} (kp)^n \cdot (1-p)^k \cdot \overbrace{\frac{1}{(1-p)^n}}^{1}$$

$$P(n) = \lim_{\substack{k \to \infty \\ p \to \langle n \rangle / k}} \frac{1}{n!} \cdot \langle n \rangle^n \cdot \left[\overbrace{(1-p)^{1/p}}^{e^{-1}} \right]^{\langle n \rangle} = \frac{1}{n!} \cdot \langle n \rangle^n \cdot e^{-\langle n \rangle}$$

Poisson distribution

Poisson distribution

Mean value

$$\sum_{n=0}^{\infty} n\cdot P(n) = \sum_{n=0}^{\infty} n\cdot\frac{1}{n!}\cdot\langle n\rangle^{n}\cdot e^{-\langle n\rangle} = \langle n\rangle\cdot e^{-\langle n\rangle}\overbrace{\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\langle n\rangle^{n-1}}^{e^{\langle n\rangle}} = \langle n\rangle$$

Quadratic moment

$$\left\langle n^{2}\right\rangle = \sum_{n=0}^{\infty} n^{2}\cdot P(n) = \sum_{n=0}^{\infty} n^{2}\cdot\frac{1}{n!}\cdot\langle n\rangle^{n}\cdot e^{-\langle n\rangle} = \sum_{n=1}^{\infty}\frac{(n-1)+1}{(n-1)!}\cdot\langle n\rangle^{n}\cdot e^{-\langle n\rangle}$$

$$\left\langle n^{2}\right\rangle = \langle n\rangle^{2}\, e^{-\langle n\rangle}\overbrace{\sum_{n=2}^{\infty}\frac{1}{(n-2)!}\cdot\langle n\rangle^{n-2}}^{e^{\langle n\rangle}} + \langle n\rangle e^{-\langle n\rangle}\overbrace{\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\cdot\langle n\rangle^{n-1}}^{e^{\langle n\rangle}} = \langle n\rangle^{2} + \langle n\rangle$$

Variance

$$\sigma^{2} \equiv \left\langle\left(n-\langle n\rangle\right)^{2}\right\rangle = \left\langle n^{2} - 2n\langle n\rangle + \langle n\rangle^{2}\right\rangle = \left\langle n^{2}\right\rangle - \langle n\rangle^{2} = \langle n\rangle$$